



**You have downloaded a document from
RE-BUS
repository of the University of Silesia in Katowice**

Title: Stability of the Pexider functional equation

Author: Roman Badora, Barbara Przebieracz, Peter Volkmann

Citation style: Badora Roman, Przebieracz Barbara, Volkmann Peter. (2010). Stability of the Pexider functional equation. "Annales Mathematicae Silesianae" (Nr 24 (2010), s. 7-13).



Uznanie autorstwa - Użycie niekomercyjne - Bez utworów zależnych Polska - Licencja ta zezwala na rozpowszechnianie, przedstawianie i wykonywanie utworu jedynie w celach niekomercyjnych oraz pod warunkiem zachowania go w oryginalnej postaci (nie tworzenia utworów zależnych).



UNIWERSYTET ŚLĄSKI
W KATOWICACH



Biblioteka
Uniwersytetu Śląskiego



Ministerstwo Nauki
i Szkolnictwa Wyższego

STABILITY OF THE PEXIDER FUNCTIONAL EQUATION

ROMAN BADORA, BARBARA PRZEBIERACZ, AND PETER VOLKMANN

Abstract. A stability result for the Pexider equation will be derived from a stability theorem published in [9] for the Cauchy functional equation. Then we discuss the quality of some constants occurring in this context; as a model case we consider functions defined on the multiplicative semigroup $\{1, 0\}$.

1. Introduction

In Theorem 1 below we describe the stability result for the Cauchy equation, which had been mentioned in the Abstract.

Let S be a groupoid, i.e., S is a set and for all $x, y \in S$ we have a product $xy \in S$. For $x \in S$ and $k = 0, 1, 2, \dots$ the powers x^{2^k} are recursively defined by

$$x^{2^0} = x^1 = x, \quad x^{2^{k+1}} = x^{2^k} x^{2^k}.$$

Józef Tabor [8] pointed out the usefulness of the following condition for stability investigations:

(T) For $x, y \in S$ there always is an entire $k \geq 1$ such that

$$(1) \quad (xy)^{2^k} = x^{2^k} y^{2^k}.$$

In the present paper, groupoids S satisfying (T) are called *Tabor groupoids*. Three examples are particular cases of them; they are ordered in decreasing generality:

Received: 3.12.2010. Revised: 22.02.2011.

(2010) Mathematics Subject Classification: 39B82, 39B52.

Key words and phrases: Pexider equation, stability, Tabor groupoids.

1. Groupoids S with a square-symmetric operation, i.e.,

$$(xy)^2 = x^2y^2, \quad x, y \in S$$

(cf. the joint paper with Zsolt Páles and R. Duncan Luce [6]). Then (1) is true for all $x, y \in S$ with the same k , viz. $k = 1$.

2. Groupoids S with a bisymmetric operation, i.e.,

$$(xy)(\bar{x}\bar{y}) = (x\bar{x})(y\bar{y}), \quad x, y, \bar{x}, \bar{y} \in S.$$

Here $\bar{x} = x$, $\bar{y} = y$ leads to square-symmetry.

3. Commutative semigroups S .

Let us mention that Zbigniew Gajda and Zygfryd Kominek [1] considered semigroups satisfying condition (T). Inspired by Józef Tabor [8], they call them weakly commutative.

Now let E be a Banach space. A subset V of E is called *ideally convex* (E. A. Lifšic [3]), if for every bounded sequence d_1, d_2, d_3, \dots in V and for every numerical sequence $\alpha_1, \alpha_2, \alpha_3, \dots \geq 0$ such that $\sum_{k=1}^{\infty} \alpha_k = 1$ we get

$$\sum_{k=1}^{\infty} \alpha_k d_k \in V.$$

The following theorem is taken from [9]; in the case of a commutative semigroup S it goes back to Jacek Tabor [7].

THEOREM 1. *Let S be a Tabor groupoid, and let V be a bounded and ideally convex subset of the Banach space E . For $f: S \rightarrow E$ we suppose*

$$f(xy) - f(x) - f(y) \in V, \quad x, y \in S.$$

Then there exists a (unique) function $F: S \rightarrow E$ such that

$$F(xy) = F(x) + F(y), \quad F(x) - f(x) \in V, \quad x, y \in S.$$

2. The Pexider equation

THEOREM 2. *Let S be a Tabor groupoid having a neutral element n , i.e., $n \in S$ and*

$$nx = xn = x, \quad x \in S.$$

Let V be a symmetric, bounded, and ideally convex subset of a Banach space E (the symmetry means $-V = V$). For $f, g, h: S \rightarrow E$ we suppose

$$(2) \quad f(xy) - g(x) - h(y) \in V, \quad x, y \in S.$$

Then there are $F, G, H: S \rightarrow E$ satisfying the Pexider equation

$$(3) \quad F(xy) = G(x) + H(y), \quad x, y \in S,$$

as well as the conditions

$$(4) \quad F(x) - f(x) \in 3V, \quad G(x) - g(x) \in 4V, \quad H(x) - h(x) \in 4V, \quad x \in S.$$

PROOF. With $y = n$ and with $x = n$ in (2) we get

$$(5) \quad f(x) - g(x) - h(n) \in V, \quad f(y) - g(n) - h(y) \in V,$$

hence $f(x) \in g(x) + h(n) + V$, $f(y) \in h(y) + g(n) + V$, thus

$$\begin{aligned} f(xy) - f(x) - f(y) + g(n) + h(n) &\in f(xy) - g(x) - h(y) + V + V \\ &\subseteq V + V + V = 3V, \end{aligned}$$

the last equality being true, since V is convex. For

$$(6) \quad \tilde{f}(x) := f(x) - g(n) - h(n), \quad x \in S,$$

this means

$$\tilde{f}(xy) - \tilde{f}(x) - \tilde{f}(y) \in 3V, \quad x, y \in S,$$

and by Theorem 1 there is a function $\Phi: S \rightarrow E$ such that

$$\Phi(xy) = \Phi(x) + \Phi(y), \quad \Phi(x) - \tilde{f}(x) \in 3V, \quad x, y \in S.$$

Now it is easily seen that for $F(x) := \Phi(x) + g(n) + h(n)$, $G(x) := \Phi(x) + g(n)$, $H(x) := \Phi(x) + h(n)$, $x \in S$, we get (3) and (4):

(3) is obvious; $F(x) - f(x) \in 3V$ follows from (6) and $\Phi(x) - \tilde{f}(x) \in 3V$; the remaining formulae in (4) are consequences of (5), (6), and $\Phi(x) - \tilde{f}(x) \in 3V$. \square

REMARK 1. Theorem 2 should be compared to other stability results for the Pexider equation, e.g. to those of Kazimierz Nikodem [5] and Zygfryd Kominek [2], where the target space for the functions is more general than a Banach space.

When choosing $V = \{x \mid x \in E, \|x\| \leq \varepsilon\}$, then we get from Theorem 2 the following Corollary, which had been obtained by Nikodem [5] in the case of a commutative semigroup S .

COROLLARY 1. *Let S be a Tabor groupoid having a neutral element, and let E be a Banach space. For $f, g, h: S \rightarrow E$ we suppose*

$$\|f(xy) - g(x) - h(y)\| \leq \varepsilon, \quad x, y \in S.$$

Then there are $F, G, H: S \rightarrow E$ such that

$$F(xy) = G(x) + H(y), \quad x, y \in S,$$

$$(7) \quad \|F(x) - f(x)\| \leq 3\varepsilon, \quad \|G(x) - g(x)\| \leq 4\varepsilon, \quad \|H(x) - h(x)\| \leq 4\varepsilon, \quad x \in S.$$

REMARK 2. If S is a commutative semigroup, then according to Zenon Moszner's survey [4], the constants 4ε in (7) can be replaced by 3ε . We do not know, whether this also holds for arbitrary Tabor groupoids.

REMARK 3. In the next paragraph we shall consider the commutative semigroup $S = \{1, 0\}$. It will follow that 3ε in (7) cannot be replaced by a number less than 2ε . It also will follow that, when having in (7) the better inequality $\|F(x) - f(x)\| \leq 2\varepsilon$, then the constants 4ε cannot be replaced by numbers less than $3\varepsilon/2$.

REMARK 4. By calculations similar to those in the next paragraph, it can be shown that for the cyclic groups $S = Z_2$, $S = Z_3$ of two and of three elements, respectively, all the numbers 3ε , 4ε in (7) can be replaced by ε .

3. The semigroup $S = \{1, 0\}$

In $S = \{1, 0\}$ we have $1 \cdot 1 = 1$, $1 \cdot 0 = 0 \cdot 1 = 0 \cdot 0 = 0$. It is easily seen that in this case solutions of the Pexider equation (3) necessarily are constant functions:

$$G(1) = G(0) = a, \quad H(1) = H(0) = b, \quad F(1) = F(0) = a + b.$$

THEOREM 3. Consider $S = \{1, 0\}$, let N be a normed space, and let $f, g, h: S \rightarrow N$ satisfy

$$(8) \quad \|f(xy) - g(x) - h(y)\| \leq \varepsilon, \quad x, y \in S.$$

Then there exist $a, b \in N$ such that

$$(9) \quad \|g(1) - a\| \leq \frac{1}{2}\varepsilon, \quad \|h(1) - b\| \leq \frac{1}{2}\varepsilon,$$

$$(10) \quad \|g(0) - a\| \leq \frac{3}{2}\varepsilon, \quad \|h(0) - b\| \leq \frac{3}{2}\varepsilon,$$

$$(11) \quad \|f(x) - a - b\| \leq 2\varepsilon, \quad x = 0, 1.$$

PROOF. Indeed, (8) means

$$(12) \quad \begin{aligned} f(1) - g(1) - h(1) &= r_1, \\ f(0) - g(1) - h(0) &= r_2, \\ f(0) - g(0) - h(1) &= r_3, \\ f(0) - g(0) - h(0) &= r_4, \end{aligned}$$

where

$$(13) \quad \|r_j\| \leq \varepsilon, \quad j = 1, 2, 3, 4.$$

We easily get

$$(14) \quad g(0) - g(1) = r_2 - r_4,$$

$$(15) \quad h(0) - h(1) = r_3 - r_4,$$

$$(16) \quad f(0) - g(1) - h(1) = r_2 + r_3 - r_4.$$

We define

$$a = g(1) + \frac{1}{2}r_2, \quad b = h(1) + \frac{1}{2}r_3,$$

then (13) already leads to (9). From (14), (15) we now get

$$(17) \quad g(0) = g(1) + r_2 - r_4 = a + \frac{1}{2}r_2 - r_4,$$

$$(18) \quad h(0) = h(1) + r_3 - r_4 = b + \frac{1}{2}r_3 - r_4,$$

and this gives (10). From (12), (16) we finally have

$$\begin{aligned} f(1) - a - b &= r_1 + g(1) - a + h(1) - b = r_1 - \frac{1}{2}r_2 - \frac{1}{2}r_3, \\ f(0) - a - b &= f(0) - g(1) - \frac{1}{2}r_2 - h(1) - \frac{1}{2}r_3 = \frac{1}{2}r_2 + \frac{1}{2}r_3 - r_4, \end{aligned}$$

and these two lines prove (11). \square

EXAMPLE. The following example shows that

- I) 2ε in (11) is best possible,
- II) having 2ε in (11), then also $\frac{1}{2}\varepsilon$ in (9) and $\frac{3}{2}\varepsilon$ in (10) are best possible:

We define $f, g, h: S \rightarrow \mathbb{R}$ by

$$f(1) = 2, \quad f(0) = -2, \quad g(1) = 1, \quad g(0) = -1, \quad h(1) = 0, \quad h(0) = -2.$$

Then

$$\begin{aligned} f(1) - g(1) - h(1) &= 2 - 1 - 0 = 1, \\ f(0) - g(1) - h(0) &= -2 - 1 + 2 = -1, \\ f(0) - g(0) - h(1) &= -2 + 1 - 0 = -1, \\ f(0) - g(0) - h(0) &= -2 + 1 + 2 = 1, \end{aligned}$$

hence (8) holds for

$$(19) \quad \varepsilon = 1$$

(with absolute value in \mathbb{R} being the norm).

PROOF OF I). Suppose (11) to hold for some $a+b \in \mathbb{R}$ and with 2ε replaced by some η :

$$(20) \quad |2 - a - b| = |f(1) - a - b| \leq \eta, \quad |-2 - a - b| = |f(0) - a - b| \leq \eta.$$

Then $4 \leq |2 - a - b| + |-2 - a - b| \leq 2\eta$, hence (cf. (19)) $2\varepsilon = 2 \leq \eta$.

PROOF OF II). Inequality (11) with $2\varepsilon = 2$ leads to (20) with $\eta = 2$, hence to $a + b = 0$, i.e., $b = -a$. Then (10) with η instead of $\frac{3}{2}\varepsilon$ leads to

$$|-1 - a| = |g(0) - a| \leq \eta, \quad |-2 + a| = |h(0) - b| \leq \eta,$$

which implies $3 = (2 - a) + (1 + a) \leq 2\eta$, hence $\frac{3}{2}\varepsilon = \frac{3}{2} \leq \eta$.

In the same way we get from (9) with $\frac{1}{2}\varepsilon$ replaced by η that

$$|1 - a| = |g(1) - a| \leq \eta, \quad |0 + a| = |h(1) - b| \leq \eta,$$

which implies $1 = (1 - a) + (0 + a) \leq 2\eta$, hence $\frac{1}{2}\varepsilon = \frac{1}{2} \leq \eta$.

Acknowledgement. We cordially thank Mrs. Marion Ewald for her type-writing of this paper. The research of the second author was supported by University of Silesia (Iterative Functional Equations and Real Analysis).

References

- [1] Gajda Z., Kominek Z., *On separation theorems for subadditive and superadditive functionals*, Studia Math. **100** (1991), 25–38.
- [2] Kominek Z., *On Hyers–Ulam stability of the Pexider equation*, Demonstratio Math. **37** (2004), 373–376.
- [3] Lifšic E.A., *Ideal’no vypuklye množestva*, Funkcional’. Analiz Priložen. **4** (1970), no. 4, 76–77.
- [4] Moszner Z., *On the stability of functional equations*, Aequationes Math. **77** (2009), 33–88.
- [5] Nikodem K., *The stability of the Pexider equation*, Ann. Math. Sil. **5** (1991), 91–93.
- [6] Páles Z., Volkman P., Luce R.D., *Hyers–Ulam stability of functional equations with a square-symmetric operation*, Proc. Nat. Acad. Sci. U.S.A. **95** (1998), 12772–12775.
- [7] Tabor Jacek, *Ideally convex sets and Hyers theorem*, Funkcial. Ekvac. **43** (2000), 121–125.
- [8] Tabor Józef, *Remark 18* (at the 22nd International Symposium on Functional Equations, Oberwolfach 1984), Aequationes Math. **29** (1985), 96.
- [9] Volkman P., *O stabilności równań funkcyjnych o jednej zmiennej*, Sem. LV, no. 11 (2001), 6 pp., Errata ibid. no. 11 bis (2003), 1 p., <http://www.math.us.edu.pl/smdk>

INSTITUTE OF MATHEMATICS
SILESIA UNIVERSITY
BANKOWA 14
40-007 KATOWICE
POLAND
e-mail: robadora@math.us.edu.pl
e-mail: barbara.przebieracz@us.edu.pl

INSTITUT FÜR ANALYSIS
KIT
76128 KARLSRUHE
GERMANY